Application of Taylor Models to the Worst-Case Analysis of Stripline Interconnects

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***Abstract*—This paper outlines a preliminary application of Taylor models to the worst-case analysis of transmission lines with bounded uncertain parameters. Taylor models are an algebraic technique that represents uncertain quantities in terms of a Taylor expansion complemented by an interval remainder encompassing approximation and truncation errors. The Taylor model formulation is propagated from input uncertainties to output responses through a suitable redefinition of the algebraic operations involved in their calculation. While the Taylor expansion defines an analytical and parametric model of the response, the remainder provides a conservative bound inside which the true value is guaranteed to lie. The approach is validated against the SPICE simulation of a coupled stripline and shows promising accuracy and efficiency.**

***Index Terms*—Parametric simulation, multiconductor transmission lines, Taylor models, tolerance analysis, uncertainty, worst-case analysis.**

I. Introduction

Owing to the increasing impact of uncertainty in modern electronic devices, extensive research studies have been conducted over efficient techniques that take the effect of variability into account. Several approaches based on polynomial chaos were investigated for the signal and power integrity analysis in high-speed interconnects [1]–[5]. These techniques allow to compute stochastic properties of propagating signals, like statistical moments and distribution functions. Hence, they are referred to as "probabilistic". The maximum and minimum signal levels that can be expected are estimated within a given probability (confidence bounds).

Sometimes, however, a more rigorous assessment of possible violation of design specifications is required. In this case, a precautionary, conservative estimation of response bounds is preferred over probabilistic measures. This is the main goal of worst case (WC) analysis [6]. In this context, Monte Carlo (MC) is commonly adopted as a design exploration tool. A MC analysis inherently provides an underestimation of WC bounds, as only a finite subset of the possible realizations is taken into account. The accuracy increases with the number of samples. However, WC responses often fall in the tails of probability distribution and have therefore a low probability to occur, thus requiring a large number samples to be captured. Polynomial chaos can be used as a faster surrogate for MC analysis, but its approximation may introduce a further underestimation of the "true" WC bounds.

Interval methods were proposed as a viable solution to propagate bounded uncertain input parameters to output responses [7]. Interval arithmetic (IA) redefines basic algebraic operators over bounded inputs with the aim of determining a conservative bound on the output. Although rather simple to implement, IA suffers from being too pessimistic and it often results in severe overestimation. Affine arithmetic has been proposed as an enhancement that has been applied to interconnect analysis [8], [9]. Nevertheless, a systematic treatment of nonlinear functions and matrix operations is unavailable or highly non-trivial.

In this regard, Taylor models (TM) represent an alternative and more rigorous approach [10]. According to the TM formulation, each variable is expressed as a Taylor polynomial expansion of selected design parameters and it is complemented by an interval remainder accounting for truncation and round-off errors. Algebraic operations are carried out via modified rules or, whenever a nonlinear operation occurs, by using the theorems of Taylor expansion and Lagrange remainder. In order to allow propagating the uncertainty from input parameters to output responses, the remainder is possibly updated at each operation to account for the approximations introduced on the polynomial part. Therefore, the combination between the bounds of the Taylor expansion and the interval remainder yields a conservative estimation of the WC response bounds. An application of TM to circuit simulations through the formalism of modified nodal analysis has been proposed in [11]. In this paper, the framework is applied for the first time to transmission line analysis.

II. Taylor Models

Consider a system depending on a d-variate set of uncertain and independent design parameters x over a bounded domain D. For the sake of convenience, the bound of each parameter is assumed to be normalized so that $D = [-1, 1]^d$. It is worth noting that this can always be achieved by properly shifting and rescaling the original variables.

The TM of a generic variable f that depends on the design parameters is defined as

$$T_f = P_f(x) + I_f, \tag{1}$$

where P_f is a polynomial representing the Taylor expansion of $f(x)$ calculated with a predefined order n around the center of the interval, $x_0 = 0$, whereas $I_f = [\underline{I_f}, \overline{I_f}]$ is an interval remainder such that $f(x) \in P_f(x) + I_f, \forall x \in D$. In other words, the true value of the variable f is guaranteed to lie between two hypersurfaces:

$$P_f(x) + \underline{I_f} \leq f(x) \leq P_f(x) + \overline{I_f}. \tag{2}$$

It immediately follows that if $I_f = [0, 0]$, the polynomial P_f coincides with f.

In fact, the polynomial part P_f defines a parametric representation of the variable f as a function of the parameters x. Furthermore, the combination of the minimum and maximum of the polynomial with the interval remainder yields an estimation of the overall bounds of f. By defining the polynomial bound operator $B(\cdot)$ as

$$B(P(x)) = [\underline{P}, \overline{P}] = \left[\min_x\{P(x)\}, \max_x\{P(x)\}\right], \tag{3}$$

it follows that

$$\underbrace{\underline{P_f} + \underline{I_f}}_{\underline{T_f}} \leq \min_x\{f(x)\} \leq f(x) \leq \max_x\{f(x)\} \leq \underbrace{\overline{P_f} + \overline{I_f}}_{\overline{T_f}}. \tag{4}$$

It is important to remark that, in the multivariate case ($d > 1$), the exact calculation of the maximum and minimum of a polynomial is often not possible, and a conservative estimation must be searched for. In [11], the conversion of Taylor polynomials into Bernstein polynomials was proposed, since Bernstein polynomials have the notable property of being bounded by the value of their largest and smallest coefficient.

By assuming that every variable is represented by a TM like (1), the addition/subtraction of two TM is readily given by

$$T_f \pm T_g = (P_f(x) \pm P_g(x)) + (I_f \pm I_g), \tag{5}$$

where the sum over the polynomials is error-free, while the operation over the remainders is carried out using the rules of IA [7]:

$$I_f + I_g = [\underline{I_f} + \underline{I_g}, \overline{I_f} + \overline{I_g}] \tag{6a}$$

$$I_f - I_g = [\underline{I_f} - \overline{I_g}, \overline{I_f} - \underline{I_g}]. \tag{6b}$$

The multiplication between two TM yields instead

$$T_f \cdot T_g = P_n(x) + P_e(x) + P_f(x)I_g + P_g(x)I_f + I_f I_g, \tag{7}$$

where $P_n(x)$ is the part of the product $P_f(x)P_g(x)$ up to order n, whilst $P_e(x)$ is the remaining higher-order contribution. In order to preserve the original expansion order n, the term $P_n(x)$ is retained as the polynomial part of the product, while all remaining quantities are encompassed into the interval remainder given by $B(P_e) + B(P_f)I_g + B(P_g)I_f + I_f I_g$, with the multiplications carried out in the IA-sense, e.g.,

$$\begin{aligned} I_f I_g = [\,&\min\{\underline{I_f}\,\underline{I_g}, \underline{I_f}\overline{I_g}, \overline{I_f}\underline{I_g}, \overline{I_f}\,\overline{I_g}\}, \\ &\max\{\underline{I_f}\,\underline{I_g}, \underline{I_f}\overline{I_g}, \overline{I_f}\underline{I_g}, \overline{I_f}\,\overline{I_g}\}]. \end{aligned} \tag{8}$$

The operations introduced so far are readily extended to complex- and matrix-valued calculations, by operating separately on the real and imaginary part or element-wise, respectively. When a nonlinear function of a scalar variable occurs instead, it is replaced by its nth-order Taylor expansion [10]. This being in polynomial form, it merely involves the calculation of additions and multiplications between TM. The output remainder is computed based on the bound of the Lagrange remainder. In the following section, the specific case of sine and cosine functions, which are involved in the solution of transmission line equations, is addressed. For the general case, the reader is referred to [11].

III. Application to Transmission Lines

The TM arithmetic is now applied to a distributed interconnect of length ℓ described by Telegraphers' equations, whose frequency-domain formulation at a given angular frequency ω is [12]

$$\frac{d}{dz}\boldsymbol{V}(z) = -j\omega\boldsymbol{L} \cdot \boldsymbol{I}(z) \tag{9a}$$

$$\frac{d}{dz}\boldsymbol{I}(z) = -j\omega\boldsymbol{C} \cdot \boldsymbol{V}(z), \tag{9b}$$

where $z \in [0, \ell]$ is the longitudinal coordinate, vectors $\boldsymbol{V}$ and $\boldsymbol{I}$ collect the voltages and currents along the line, respectively, and $\boldsymbol{L}$ and $\boldsymbol{C}$ are the per-unit-length inductance and capacitance matrices describing the electromagnetic propagation and coupling. For the sake of simplicity, in this paper the line is assumed to be lossless and lying in a homogeneous dielectric medium with relative permittivity ε_r.

The kernel in the solution of (9) is the calculation of the chain-parameter matrix which, under the aforementioned assumptions, simplifies to the four blocks given by

$$\boldsymbol{\Phi}_{11} = \cos(\beta\ell)\boldsymbol{1} \tag{10a}$$

$$\boldsymbol{\Phi}_{12} = -j\frac{c_0}{\sqrt{\varepsilon_r}}\sin(\beta\ell)\boldsymbol{L} \tag{10b}$$

$$\boldsymbol{\Phi}_{21} = -j\sqrt{\varepsilon_r}c_0\sin(\beta\ell)\boldsymbol{C}_0 \tag{10c}$$

$$\boldsymbol{\Phi}_{22} = \cos(\beta\ell)\boldsymbol{1}, \tag{10d}$$

where c_0 the speed of light in vacuum, $\boldsymbol{C}_0$ is the capacitance matrix computed in vacuum (i.e., with $\varepsilon_r = 1$), $\boldsymbol{1}$ is the identity matrix of the same size as $\boldsymbol{L}$ and $\boldsymbol{C}_0$, and $\beta = \sqrt{\varepsilon_r}\,\omega/c_0$ is the propagation constant.

The equations (10) involve the sine and cosine functions, which need to be properly handled within the considered TM framework. According to [10], the calculation of a nonlinear function $g(\cdot)$ of a TM is carried out by considering its functional Taylor expansion

$$g(T_f) = P_g(x) + I_g = \sum_{k=0}^{n} \frac{g^{(k)}(c_f)}{k!}(T_f - c_f)^k + \frac{B(g^{(n+1)}(\zeta))}{(n+1)!} B((T_f - c_f)^{n+1}) \tag{11}$$

centered at $c_f = P_f(0)$. The second term in the r.h.s. is the bound of the Lagrange remainder, with $\zeta \in [\underline{T_f}, \overline{T_f}] = B(T_f)$. The coefficients $g^{(k)}(c_f)$ are the functional derivatives of g calculated at c_f and, for the specific case of sine and cosine functions, they are

$$g^{(k)}(c_f) = \begin{cases} (-1)^{\frac{k}{2}} \sin(c_f) & k \text{ even} \\ (-1)^{\frac{k-1}{2}} \cos(c_f) & \text{otherwise} \end{cases} \tag{12}$$

and

$$g^{(k)}(c_f) = \begin{cases} (-1)^{\frac{k}{2}} \cos(c_f) & k \text{ even} \\ (-1)^{\frac{k-1}{2}+1} \sin(c_f) & \text{otherwise,} \end{cases} \tag{13}$$

respectively.

It is worth noting that the first term in the r.h.s. of (11) merely involves additions and multiplications of TM, which are computed with the rules outlined in Section II. Furthermore, the bounds of the Lagrange remainder linearly depend on the bounds of the nonlinear function $B(g^{(n+1)}(\zeta))$, which in turn requires the calculation of the bounds $B(\sin(\zeta))$ or $B(\cos(\zeta))$ in the pertinent domain of ζ. The bound estimation is complicated by the periodic behavior of the sine and cosine functions. Since their value is in any case limited within $[-1, 1]$, a naive solution is to overestimate these bounds by assuming $B(\sin(\zeta)) = B(\cos(\zeta)) = [-1, 1]$. However, tighter bounds can be obtained by considering the function as locally monotonic and by including the effect of a possible slope change only when a stationary point belongs to the domain of ζ. This leads to

$$\min\{\sin(\zeta)\} = \begin{cases} -1 & \frac{3\pi}{2} + 2\pi k \in B(T_f) \\ \min\{\sin(\underline{T}_f), \sin(\overline{T}_f)\} & \text{otherwise} \end{cases} \tag{14a}$$

$$\max\{\sin(\zeta)\} = \begin{cases} +1 & \frac{\pi}{2} + 2\pi k \in B(T_f) \\ \max\{\sin(\underline{T}_f), \sin(\overline{T}_f)\} & \text{otherwise} \end{cases} \tag{14b}$$

$$\min\{\cos(\zeta)\} = \begin{cases} -1 & \pi + 2\pi k \in B(T_f) \\ \min\{\cos(\underline{T}_f), \cos(\overline{T}_f)\} & \text{otherwise} \end{cases} \tag{14c}$$

$$\max\{\cos(\zeta)\} = \begin{cases} +1 & 2\pi k \in B(T_f) \\ \max\{\cos(\underline{T}_f), \cos(\overline{T}_f)\} & \text{otherwise,} \end{cases} \tag{14d}$$

with $k \in \mathbb{Z}$.

Finally, once the TM of the chain-parameter matrix is available, the voltages and currents at the line termination are readily computed by incorporating the information on the loading conditions. Assuming the Thévenin equivalent $\boldsymbol{V}(0) = \boldsymbol{V}_S - \boldsymbol{Z}_S \boldsymbol{I}(0)$ at the source, and a load admittance $\boldsymbol{Y}_L$, this leads to [12]

$$\boldsymbol{I}(0) = (\boldsymbol{\Phi}_{21}\boldsymbol{Z}_S + \boldsymbol{Y}_L\boldsymbol{\Phi}_{12} - \boldsymbol{\Phi}_{22} - \boldsymbol{Y}_L\boldsymbol{\Phi}_{11}\boldsymbol{Z}_S)^{-1} \times (\boldsymbol{\Phi}_{21} - \boldsymbol{Y}_L\boldsymbol{\Phi}_{11})\boldsymbol{V}_S \tag{15}$$

$$\boldsymbol{V}(0) = \boldsymbol{V}_S - \boldsymbol{Z}_S\boldsymbol{I}(0) \tag{16}$$

$$\boldsymbol{V}(\ell) = \boldsymbol{\Phi}_{11}\boldsymbol{V}_S + (\boldsymbol{\Phi}_{12} - \boldsymbol{\Phi}_{11}\boldsymbol{Z}_S)\boldsymbol{I}(0) \tag{17}$$

$$\boldsymbol{I}(\ell) = \boldsymbol{Y}_L\boldsymbol{V}(\ell). \tag{18}$$

The matrix inversion in (15) is performed with the algorithm described in [11].

IV. Results and Validation

The proposed technique is implemented in MATLAB and it is applied to the analysis of the coupled stripline of Fig. 1 [3]. All the relevant information about the line geometry, material and terminations is indicated in the figure. The per-unit-length inductance and capacitance matrices are computed in SPICE and are

$$\boldsymbol{L} = \begin{bmatrix} 423.09 & 93.32 \\ 93.32 & 423.09 \end{bmatrix} \text{nH/m}$$

$$\boldsymbol{C} = \varepsilon_r \begin{bmatrix} 27.64 & -6.10 \\ -6.10 & 27.64 \end{bmatrix} = \begin{bmatrix} 110.57 & -24.39 \\ -24.39 & 110.57 \end{bmatrix} \text{pF/m}$$

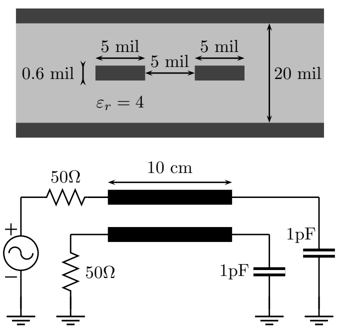

Fig. 1. Cross-section (top) and terminations (bottom) of the coupled stripline.

Fig. 2 shows the voltage at the far-end termination of the active line (top panel), as well as the far-end crosstalk (bottom panel), computed for a parametric sweep of the relative permittivity in the interval $[3.8, 4.2]$. The results from a parametric SPICE simulation (solid blue lines) are compared against the model provided by the polynomial part of a third-order TM of the load voltages (dashed red lines) computed with the proposed technique. An excellent agreement is established.

Next, the value of the load capacitors is considered as an additional uncertain variable in the range $[0.95, 1.05]$ pF. A WC analysis is carried out to assess the upper and lower bounds of the far-end voltages, resulting from the uncertainty on the relative permittivity of the stripline and the load capacitance. A reference result is calculated in SPICE by means of a MC analysis with 10000 samples. The spread of the responses is indicated as a gray area in Fig. 3. The response bounds are also estimated with a third-order TM, by combining the upper and lower bounds of the polynomial part

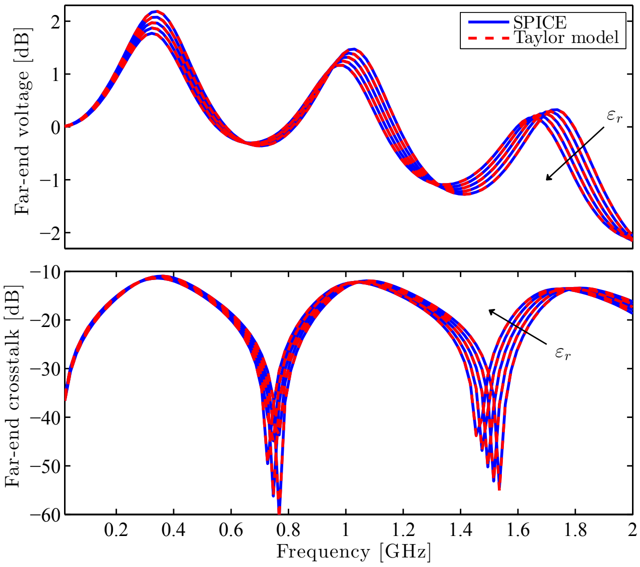

Fig. 2. Voltages at the far-end termination of the active (top panel) and quiet (bottom panel) stripline conductors, computed for a parametric sweep of the relative permittivity in SPICE (solid blue lines) or with a third-order TM (dashed red lines).

with the interval remainder, and they are plotted as blue lines. The comparison highlights a generally very good performance of the TM in capturing the upper and lower bounds of the response, with a slight loss of accuracy around the second resonance.

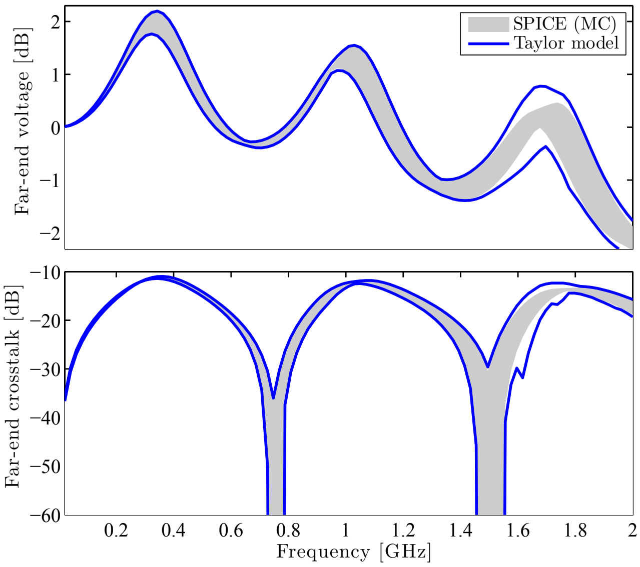

Fig. 3. Spread of the far-end voltages resulting from parameter uncertainty (gray areas). The blue lines are the upper and lower bounds predicted by a third-order TM.

As far as the computational times are concerned, the MC simulation in SPICE takes 680 s, whereas the TM calculation requires 19 s for the univariate model (Fig. 2) and 35.6 s for the bivariate model (Fig. 3). A speed-up between 19× and 36× is therefore achieved over a MC analysis, while obtaining in addition a parametric model of the responses.

V. CONCLUSIONS

This paper presents an application of TM to the WC analysis of transmission line responses. The method represents all the quantities that depend on selected uncertain design parameters as a Taylor polynomial expansion complemented by an interval remainder accounting for approximation and round-off errors. The TM representation is propagated from input parameters to output responses via a suitable redef nition of the algebraic operations involved in their calculation. The polynomial part provides an accurate parametric model of the transmission line response. Furthermore, the combination of the polynomial bounds with the interval remainder allows for a guaranteed-conservative estimation of the WC bounds. The technique is applied to the analysis of a coupled stripline and shows promising results in terms of both accuracy and eff ciency.

ACKNOWLEDGMENT

This work was supported by the Research Foundation Flanders (FWO-Vlaanderen), of which Paolo Manfredi is a Post-Doctoral Research Fellow.

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